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HEAT TRANSFER IN A ROTATING CYLINDER OF FLUID HEATED FROM ABOVE

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Abstract—We present a mathematical analysis of centrifugally driven thermal convection in a cylinder which rotates about its vertical axis. Two extensions are made of previous work. First, a heat flux through the side wall of the cylinder is considered and its effect on the Nusselt number for the top and bottom cylinder surfaces is determined. Secondly, a constant heat flux boundary condition for the horizontal surface is investigated.

NOMENCLATURE

a,	cylinder radius;
A,	$g/\omega^2 a$, the acceleration ratio;
c_n,	expansion coefficients in (23);
g,	acceleration of gravity;
h,	cylinder half-height;
$\boldsymbol{i}$,	unit vector in the radial direction;
I_0,	modified Bessel function of the first kind of order zero;
k,	fluid thermal conductivity;
$\boldsymbol{k}$,	unit vector in the positive z direction;
K,	temperature difference ratio, defined in (14);
$\mathscr{L}_\gamma^2$,	two dimensional second order operator defined in (9),
Nu,	Nusselt number,
Nu_{ex}	external Nusselt number governing heat flux through cylinder side wall;
p,	fluid pressure;
$\boldsymbol{q}$,	fluid velocity vector;
Q,	heat flux per unit area;
r,	radial coordinate;
r^*	r/a;
s,	expansion parameter for constant heat flux case, in (60);
T,	temperature;
ΔT,	$(T_a - T_b)/2$;
T^*,	$(T - T_0)/\Delta T$ section 1; $(Qh/k)(T - T_0)$ section 2;
u,	radial velocity;
v,	tangential velocity;
v^*,	$v\dfrac{2}{\alpha\Delta T\omega a}$ section 1; $v\dfrac{2k}{\omega\alpha Qha}$ section 2;
w,	axial velocity;
z,	axial coordinate;
z^*,	z/h;
α,	coefficient of thermal expansion;
β,	$\alpha\Delta T/4$, the thermal Rossby number;
β^*	$\alpha Qh/4k$;
γ,	a/h, the aspect ratio;
ε,	$\nu/2\omega h^2$, the Ekman number;
θ,	dimensionless temperature;
κ,	fluid thermal diffusivity;
λ,	$\sigma\beta/\varepsilon^{\frac{1}{2}}$, controlling parameter;
λ^*,	$\sigma\beta^*/\varepsilon^{\frac{1}{2}}$;
ν,	fluid kinematic viscosity;
ρ,	fluid density;
σ,	ν/κ, the Prandtl number;
$\Phi(r,z)$,	correction to conduction profiles;
$\Phi_n(r)$, $\chi_n(z)$,	expansion functions for $\Phi(r,z)$;

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ψ, stream function defined by $w = -r^{-1}\psi_r$ and $u = r^{-1}\psi_z$.
ω, rotational frequency.

Subscripts
a, top conditions;
b, bottom conditions;
0, reference conditions;
0, 1, 2, expansion indices;
∞, ambient conditions.

Superscripts
$'$, reduced quantity (pressure);
$*$, dimensionless quantities.

INTRODUCTION

If a temperature difference is imposed normal to an acceleration in a fluid, motion will be produced. The most common example of this, of course, is free convection in which the acceleration in question is that of gravity. The centrifugal acceleration in a rotating fluid can have a similar effect. Thus consider a rotating fluid which would be in solid body rotation if it were isothermal. Now impose a temperature difference normal to the centrifugal acceleration; motion relative to the solid body rotation will be induced and this motion will produce an augmentation of heat transfer over that which would occur in the absence of convection. The centrifugal acceleration can be many times greater than that of gravity, and it is of interest to determine under what conditions the effected heat fluxes are large.

It was shown by Schmidt that rotation could augment heat transfer in water cooled turbine blades [13]. More recently, several studies have been made of convection in rotating tubes [10–12] and in a closed loop rotating thermosyphon [2].

Lemlich and co-workers have made an extensive study of the convection near a flat plate which rotates in a synchronously rotating surroundings [6. 7. 9]; the axis of rotation is in the plane of the plate. As pointed out by Manoff and Lemlich [9], it is important to differentiate between their work in which rotation is around an axis in the plate and work to be discussed below in which the axis of rotation is normal to the flat surface. It is shown in [9] that the Coriolis acceleration has a very small effect on the rate of heat transfer to the rotating plate.

We have recently analyzed centrifugally driven thermal convection in a fluid contained between two horizontal disks of infinite radius [5]. The disks rotate at the same angular velocity about a common axis of rotation which is normal to both the disks. The top disk is heated relative to the bottom disk. A characteristic rotational Reynolds number is large such that a boundary layer regime occurs. A balance between the centrifugal and Coriolis forces produces a strong tangential circulation. However, a secondary flow produced by Ekman (boundary) layers on the two disks makes large heat transfer rates possible.

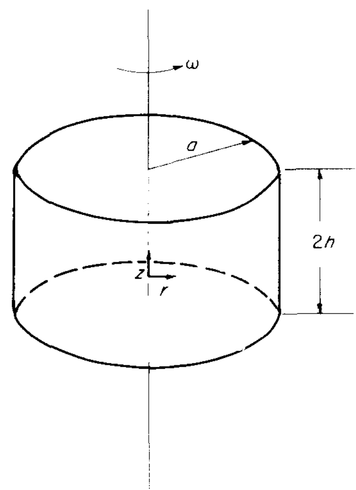

Fig. 1. The rotating cylinder.

In [3], we extended the analysis to a radially bounded system. The fluid is now contained in a cylinder of radius a and height $2h$ which rotates about its vertical axis at a constant angular velocity ω (Fig. 1). The top surface is again heated relative to the lower surface; each is isothermal. The curved side wall was taken to be either perfectly insulated (no heat loss at any height) or perfectly conducting. (By perfectly conducting we mean that the temperature of the side wall varies linearly between that of the hot upper surface and the cool lower surface.) It was found that the presence of the side wall had a significant effect on the rate of heat transfer from the upper to the lower surfaces. Secondly, it is shown that the type of boundary condition imposed at the side wall has a substantial effect on the top and bottom plate Nusselt numbers. (Much greater than would be expected from experience with non-rotating natural convection.) It is seen that any analysis based on a radially unbounded system may or may not be the limit of any physically realizable experiment. In fact, it is shown in [4] that the Nusselt numbers of the top and bottom surfaces of a cylinder with a perfectly conducting side wall approach the Nusselt numbers for the radially unbounded system [5] as the cylinder radius is increased, but the Nusselt numbers for the cylinder with insulated side walls do not.

We have also been carrying out an experimental investigation of the rotating cylinder described above [1, 8]. In order to facilitate a comparison between the experimental and mathematical results, we have extended the analysis of [3] and this extension is the contents of the present paper. We consider two features. First, in the experimental apparatus the side wall of the cylinder is made of acrylic plastic and heavily covered with insulation to minimize heat loss. It is important to know the effect of side wall heat losses on the Nusselt numbers of the top and bottom disks since as shown in [3] the boundary condition at the side wall has a large effect; thus we determine below the dependence of the Nusselt numbers on a known side wall heat transfer coefficient. In the second part of this paper, we consider a constant heat flux boundary condition of the top and bottom disks rather than the isothermal condition considered in [3].

TOP AND BOTTOM SURFACES ISOTHERMAL, HEAT LOSSES OUT SIDE WALL

Consider a right circular cylinder of height $2h$ and radius a rotating about its vertical axis with a constant angular velocity ω. A Newtonian fluid is enclosed in the cylinder. The top and bottom surface temperatures are T_a and T_b respectively where $\Delta T = (T_a - T_b)/2 > 0$. It is assumed that the resultant flow is steady, laminar and axisymmetric. The analysis is restricted to the case where a characteristic Reynolds number is large such that boundary layers form on all the cylinder surfaces. We assume that the density is constant except when multiplying the gravitational or centrifugal accelerations where we take

$$\rho = \rho_0[1 - \alpha(T - T_0)]. \tag{1}$$

The subscript denotes a reference state and α is the coefficient of thermal expansion. All other fluid properties are assumed constant.

The equations of continuity and motion are then

$$\nabla \cdot \boldsymbol{q} = 0 \tag{2}$$

$$\begin{aligned}\boldsymbol{q} \cdot \nabla \boldsymbol{q} + 2\omega(\boldsymbol{k} \times \boldsymbol{q}) + \omega r^2 \alpha (T - T_0)\boldsymbol{i} \\ - g\alpha(T - T_0)\boldsymbol{k} = -\rho_0^{-1}\nabla p' + \nu\nabla^2 \boldsymbol{q}\end{aligned} \tag{3}$$

where $\boldsymbol{k}$ and $\boldsymbol{i}$ are unit vectors in the vertical and radial directions respectively and p' is defined by

$$p' = p + z\rho_0 g - \tfrac{1}{2}\rho_0\omega^2 r^2. \tag{4}$$

We introduce a stream function into (3), cross differentiate and subtract the r and z components to eliminate p', and substitute the dimensionless quantities

$$\left.\begin{aligned} & r^* = r/a, \quad z^* = z/h, \\ & T^* = \frac{T = T_0}{\Delta T}, \quad \Delta T = (T_a - T_b)/2, \\ & \qquad\qquad T_0 = (T_a + T_b)/2, \\ & v^* = v\left(\frac{2}{\alpha\Delta T\omega a}\right), \quad \psi^* = \psi\left(\frac{2}{\alpha\Delta T\omega a^2 h}\right). \end{aligned}\right\} \tag{5}$$

This gives

$$\beta\left[r^{-1}\Psi_z[(r^{-1}\mathscr{L}_\gamma^2\psi)_r - r^{-2}\mathscr{L}_\gamma^2\psi] - r^{-2}\psi_r(\mathscr{L}_\gamma^2\psi)_z - \frac{2vv_z}{r}\right] - v_z + rT_z + A\gamma^{-1}T_r = \varepsilon r^{-1}\mathscr{L}_\gamma^4\psi \quad (6)$$

$$\beta\left[r^{-2}\frac{\partial(\psi,rv)}{\partial(z,r)}\right] + r^{-1}\psi_z = \varepsilon r^{-1}\mathscr{L}_\gamma^2(rv). \quad (7)$$

All quantities in (6) and (7) are dimensionless. The stars have been omitted for ease of writing. The nomenclature is the same as used in [3]. The dimensionless energy equation is

$$\sigma\beta\left[r^{-1}\frac{\partial(\psi,T)}{\partial(z,r)}\right] = \varepsilon\nabla_\gamma^2 T. \quad (8)$$

The operators $\mathscr{L}_\gamma^2$ and ∇_γ^2 are defined by

$$\mathscr{L}_\gamma^2 = \gamma^{-2}r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$\nabla_\gamma^2 = \gamma^{-2}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (9)$$

The subscripts r and z denote differentiation with respect to that variable.

There are thus five dimensionless parameters which so far have arisen:

$\varepsilon = \nu/2\omega h^2$ (the Ekman number),
$\sigma = c_p\mu/k$ (the Prandtl number),
$\beta = \frac{1}{4}\alpha\Delta T$ (the thermal Rossby number),
$\gamma = a/h$ (the aspect ratio),
$A = g/\omega^2 a$ (the acceleration ratio).

We assume ε, A and β to be small and σ and γ to be order one. Another grouping, $\lambda \equiv \sigma\beta\varepsilon^{-\frac{1}{2}}$, is of interest. If λ is small, conduction dominates over convection as a mechanism for transferring heat in the region between the Ekman layers on the horizontal surfaces; the Nusselt number is then not much greater than one. As λ increases, so does the Nusselt number. We thus treat the situation where $\lambda = O(1)$ and convection is an important mechanism for heat transfer.

The boundary conditions for the velocities are

$$v = \psi = \frac{\partial\psi}{\partial n} = 0 \quad (10)$$

on all solid surfaces where n is the outward normal. For the temperatures of the top and bottom surfaces there is

$$T = \pm 1, \quad z = \pm 1. \quad (11)$$

The heat transferred to the surroundings at the side wall is assumed to be given by a dimensional relation of the form

$$-k\frac{\partial T}{\partial r} = U(T - T_\infty), \quad r = a. \quad (12)$$

Here U is some overall heat transfer coefficient at the side wall, T_∞ is the ambient temperature far from the wall, (room temperature, say), k is the thermal conductivity of the fluid, and T is the fluid temperature. We put (12) in dimensionless form by setting, as before,

$$T^* = \frac{(T - T_0)}{\Delta T}, \quad \Delta T = (T_a - T_b)/2,$$
$$T_0 = (T_a + T_b)/2, \quad r^* = r/a \quad (13)$$

and introduce the parameters,

$$Nu_{ex} = Uh/k$$
$$K = (T_0 - T_\infty)/\Delta T. \quad (14)$$

Nu_{ex} is an "external" Nusselt number, specified *a priori*. Thus (12) takes the dimensionless form (dropping the stars),

$$\frac{\partial T}{\partial r} = -\gamma Nu_{ex}(T + K). \quad (15)$$

We note that for $T_\infty \cong T_b$, there is $K \cong 1$; for $T_\infty \cong T_a$, there is $K \cong -1$. Therefore, a realistic range for K is

$$-1 \leqslant K \leqslant 1. \quad (16)$$

The only difference between the problem treated in [3] and that considered here is the boundary condition at $r = 1$. In [3] we showed that by means of a boundary layer analysis that the governing equations could be reduced to

$$-\sqrt{2}\lambda\theta_{0,z} = \nabla_\gamma^2\theta_0 \quad (17)$$

where θ_0 is the leading term in a series expansion for the overall temperature; this equation still holds here. The boundary conditions at the horizontal surfaces are

$$\theta_0 = \pm 1, \quad z = \pm 1. \tag{18}$$

For perfectly insulated side walls the boundary condition at $r = 1$ is

$$\theta_{0,r} = -\frac{\lambda\gamma^2}{\sqrt{2}}\theta_{0,z}, \quad r = 1 \tag{19}$$

under the restriction $A \gg \varepsilon^{\frac{1}{6}}$ [3]. For $A \ll \varepsilon^{\frac{1}{6}}$ the boundary condition at $r = 1$ is non-linear. However, the results obtained using the linear and non-linear boundary conditions are not greatly different. We therefore restrict our attention in this paper to the simpler linear boundary condition. When heat is allowed to be transferred through the side wall the boundary condition is [from (15)]

$$\theta_{0,r} + \frac{\lambda\gamma^2}{\sqrt{2}}\theta_{0,z} = -\gamma Nu_{ex}(\theta_0 + K). \tag{20}$$

This, of course, reduces to (19) when Nu_{ex} is zero.

Equation (17) with (18) and (20) is solved by expanding in an infinite series

$$\theta_0 = z + \sum_{n=1}^{\infty} \Phi_n(r)\,\chi_n(z) \tag{21}$$

where

$$\left.\begin{aligned} &\chi_p = \exp(-\lambda z/\sqrt{2})\sin[p\tfrac{1}{2}\pi(z+1)] \\ &\beta_p^2 = \lambda^2/2 + (\tfrac{1}{2}\pi p)^2 \quad p = 1,2,\ldots \\ &1 = \sum_{n=1}^{\infty} a_n\chi_n(z) \end{aligned}\right\} \tag{22}$$

and the χ_p form a complete set on $(-1,1)$ and are orthonormal with weighting function $\exp(\sqrt{(2)}\lambda z)$. The $\Phi_n(r)$ are given by

$$\Phi_n(r) = c_n I_0(\beta_n\gamma r) + \frac{\sqrt{(2)}\lambda a_n}{\beta_n^2}. \tag{23}$$

Since at this point the respresentation (21)–(23) satisfies the differential equation (17) and the isothermal conditions (18), the only unknowns are the constants c_n. These are determined by the remaining boundary condition, (19). The series (21) is substituted into (19), and the result is multiplied by $\exp(\sqrt{(2)}\lambda z)\,\chi_m(z)$ and integrated from $z = -1$ to $z = +1$. The resulting linear algebraic equations for the c_n were truncated at N terms and solved by standard matrix methods.

Given in Table 1 are Nusselt numbers for a few representative cases. The Nusselt numbers for the top and bottom plates are given by

$$Nu(\pm 1) = 2\int_0^1 \left.\frac{\partial\theta}{\partial z}\right|_{z=\pm 1} r\,\mathrm{d}r. \tag{23a}$$

The Nusselt number is the ratio of the total heat transferred to that which would be transferred by conduction only. These results were obtained by inserting the series representation for θ into (23a), integrating, and numerically summing the resulting series. The number of terms taken was between 100 and 125. No attempt at a complete 4-parameter study (λ, γ, Nu_{ex}, K) was made, but effects of Nu_{ex} and K on the previous results were noted. (Previous cases for $Nu_{ex} = 0$ are included here for reference.) The first set given is for $\lambda = 0{\cdot}0$, (no convection), and serves to point out the symmetry in K for this case, viz.,

$$Nu(\pm 1, K) = Nu(\mp 1, -K), \quad \lambda = 0. \tag{24}$$

These calculations also assure us of the reliability of subsequent results.

A number of runs were made for $Nu_{ex} = 1{\cdot}0$. (It is noted that from (14) it can be seen that Nu_{ex} will not exceed 1·0 for almost all experimental conditions. $Nu_{ex} = 10$ is considered below, but this is an extreme case.) The relevant results in Table 1 show the following: When $K = 0$ (T_0 near T_∞) and $\lambda = 0{\cdot}1$. the Nusselt numbers for the horizontal surfaces found for $Nu_{ex} \neq 0$ do not differ greatly from those for $Nu_{ex} \mp 0$; this does not hold for $\lambda = 0{\cdot}5$. For the smaller λ this is the case because the cold fluid leaving the bottom Ekman layer loses heat to the surroundings for approximately half of the distance up the side wall [$(T_{\text{fluid}} - T_\infty) > 0$], and gains heat from the surroundings for

Table 1. Nusselt numbers for external radial heat transfer

λ	γ	Nu_{ex}	K	$Nu(+1)$	$Nu(-1)$
0·0	2·0	1·0	1·0	1·883	0·647
			0·0	1·265	1·265
			−1·0	0·647	1·883
0·1	10·0	0·0	—	1·046	1·046
		1·0	0·0	1·037	1·115
		1·0	1·0	1·199	1·016
		0·1	1·0	1·061	1·040
		0·1	0·0	1·036	1·054
		0·1	−1·0	1·012	1·067
0·5	1·0	0·0	—	1·050	1·050
		1·0	1·0	2·66	0·524
		1·0	0·0	1·500	1·616
		1·0	−1·0	0·342	2·700
		10·0	0·0	3·09	3·27
0·01	13·6	0·1	1·0	1·019	0·993
0·05				1·0322	1·012
0·10				1·063*	1·056
0·20				1·11†	1·196
0·50				—	1·946*

* Poor convergence
† No convergence.

approximately the upper half of this wall. Thus, at the top, the temperature contrast between the top plate and a fluid element entering the top Ekman layer is (for $Nu_{ex} = 1{\cdot}0$) approximately what it was when no external radial heat transfer was allowed. Thus the Nusselt numbers are approximately the same in the two cases. For higher λ, convective effects rearrange the isotherms near the side, and the above argument fails.

For $K \cong +1{\cdot}0$, $(T_b \cong T_\infty)$, the fluid traveling up the side wall loses heat to the surroundings for approximately the entire distance. The effect then is a larger temperature difference between the fluid and the top wall with a corresponding increase in $Nu(+1)$. Since heat was lost out the side, $Nu(-1)$ must drop in value to satisfy a total heat balance. For $K \cong 1{\cdot}0$, $(T_a \cong T_\infty)$, the situation is reversed. Heat is gained through the side walls, since $(T_{fluid} - T_\infty) < 0$, with a corresponding decrease in $Nu(+1)$ and increase in $Nu(-1)$.

Also in Table 1 is a case for $Nu_{ex} = 10{\cdot}0$, $K = 0$. The Nusselt numbers at $z = +1$ and $z = -1$ for this case are approximately equal, but are roughly twice as large as the corresponding insulated case. That they are nearly equal follows from the fact that for $K = 0$, heat is gained and lost over the bottom and top halves of the side walls, respectively. The total loss (or gain) over the side is nearly zero; hence, $Nu(+1) \simeq Nu(-1)$. The fact that they are larger than the corresponding insulated case can be explained as follows. Consider a fluid element on its way up the side wall, starting near the middle $z \cong 0{\cdot}0$. As it moves toward the top, its (dimensionless) temperature increases toward the top temperature $(+1{\cdot}0)$. However, it also begins to simultaneously lose large amounts of heat out the side wall ($Nu_{ex} \gg 1{\cdot}0$), which tends to lower its temperature; the net result of these opposing effects is to increase the temperature difference (and hence, the heat transfer), between the top plate and the fluid element as it enters the top Ekman layer. Thus, the Nusselt number is larger than in the case of a similar fluid element which did not lose any heat on its way up the top half of the side wall.

The results of this analysis of interest to the experimentalist indicate that to adequately approximate insulated walls, the restrictions

$$\left.\begin{array}{l} Nu_{ex} < 0(1) \\ K \cong 0{\cdot}0 \\ \lambda < 0{\cdot}1 \end{array}\right\} \quad (25)$$

or

$$\left.\begin{array}{l} Nu_{ex} \ll 1{\cdot}0 \\ K \neq 0 \\ \lambda = 0(1) \end{array}\right\} \quad (26)$$

should be met.

SPECIFIED HEAT FLUX AT TOP AND BOTTOM SURFACES; INSULATED SIDE WALLS

We consider here the same rotating cylinder (radius a, height $2h$), shown in Fig. 1, but assume that the sides are insulated, and that the heat flux per unit area, Q, to the top and bottom is specified. It is assumed that Q is independent of radial position. The basic governing equations (2)–(4), of course, remain valid, and the only modification occurs in the thermal conditions at the horizontal surfaces. In making the equations dimensionless, we use Q in a definition of a characteristic temperature, since no temperature (other than T_0, which is actually a free parameter), appears in the dimensional statement of the problem.

The dimensional thermal condition is written,

$$k\frac{\partial T}{\partial z} = +Q, \quad z = \pm h. \quad (27)$$

Thus, if we pick a dimensionless temperature (denoted by a star)

$$T^* = \frac{Qh}{k}(T - T_0) \quad (28)$$

and as before

$$z^* = z/h \quad (29)$$

(27) becomes

$$\frac{\partial T^*}{\partial z^*} = 1, \quad z^* = \pm 1. \quad (30)$$

In the equations of motion, we pick a thermal scale for the velocities which balances the terms $2\omega v_z$ and $\omega^2 r\alpha T_z$ (Coriolis and centrifugal buoyancy), which yields

$$v^* = v\left(\frac{2k}{\omega\alpha Qha}\right). \quad (31)$$

Comparison with the previous scale for isothermal boundary conditions (5),

$$v^* = v\left(\frac{2}{\alpha\Delta T\omega a}\right) \quad (32)$$

shows that the formulation of the dimensionless equations will become identical if ΔT is replaced by Qh/k. Thus, the dimensionless equations become identical to (6)–(8), if β is replaced everywhere by

$$\beta^* = \frac{aQh}{4k}. \quad (33)$$

Due again to the smallness of α, for most situations $\beta^* \ll 1$, and we seek solutions to the convection problem for the same range as before, namely

$$\sigma\beta^* = 0(\varepsilon^{-\frac{1}{2}}). \quad (34)$$

The approach is then quite similar to the previous case. For $\beta^* \ll 1$, we attempt to solve the dimensionless equations (dropping the stars on variables),

$$-v_z + rT_z + A\gamma^{-1}T_r = \varepsilon r^{-1}\mathscr{L}_\gamma^4\psi \quad (35)$$

$$\psi_z = \varepsilon\mathscr{L}_\gamma^2(rv), \quad (36)$$

$$\sigma\beta^* r^{-1}\frac{\partial(\psi,T)}{\partial(z,r)} = \varepsilon\nabla_\gamma^2 T. \quad (37)$$

The boundary conditions are now

$$\frac{\partial T}{\partial z} = 1, \quad z = \pm 1 \quad (38)$$

$$\frac{\partial T}{\partial r} = 0, \quad r = 1 \quad (39)$$

together with the dynamical conditions

$$v = \psi = \frac{\partial\psi}{\partial n} = 0 \quad (40)$$

on solid boundaries. Again we assume A, $\varepsilon \ll 1$, and seek a boundary layer solution. Away from solid boundaries, we have overall components of velocity and temperature which satisfy,

$$v_z = r\theta_z \tag{41}$$

$$\psi_z = 0 \tag{42}$$

$$-\sigma\beta^* \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z} = \varepsilon \nabla_\gamma^2 \theta. \tag{43}$$

Our first step is to construct Ekman layers and corrections to v, ψ, and θ within these layers. Corrections to θ are $O(\varepsilon)$ here, so θ must satisfy

$$\frac{\partial \theta}{\partial z} = 1, \quad z = \pm 1 \tag{44}$$

by itself. The Ekman layer solutions necessary to reduce v to zero at $z = \pm 1$ are easily found as in [3] with the result that the Ekman suction velocity is again $O(\varepsilon^{\frac{1}{2}})$. Equations (41) and (42) integrate simply to give

$$v(r,z) = \theta(r,z) + h(r) \tag{45}$$

$$\psi = \psi(r). \tag{46}$$

Then in the usual manner, the Ekman suction condition that the axial velocities must match will yield $h(r)$ and $\psi(r)$. This condition is

$$\psi(r)\Big|_{z=\pm 1} = \pm \frac{\varepsilon^{\frac{1}{2}}}{\sqrt{2}} rv\Big|_{z=\pm 1}. \tag{47}$$

Use of (47) gives

$$h(r) = \frac{r^2[\theta_t(r) + \theta_b(r)]}{2}, \tag{48}$$

$$\psi(r) = \frac{\varepsilon^{\frac{1}{2}}}{2\sqrt{2}} r^2[3\theta_t(r) + \theta_b(r)] \tag{49}$$

In (48) and (49), $\theta_t(r)$ and $\theta_b(r)$ denote the plate temperatures at $z = \pm 1$ respectively, viz.

$$\theta_t(r) = \theta(r, +1), \qquad \theta_b(r) = \theta(r, -1). \tag{50}$$

These are unknown in the present problem. Thus, we have v and ψ expressible in terms of the unknown θ. This function must satisfy

$$-\frac{\lambda^*}{2\sqrt{2}}\frac{1}{r}\frac{d}{dr}[r^2(3\theta_t + \theta_b)]\frac{\partial \theta}{\partial z} = \nabla_\gamma^2 \theta \tag{51}$$

$$\frac{\partial \theta}{\theta z} = 1, \quad z = \pm 1, \tag{52}$$

where

$$\lambda^* = \sigma\beta^*/\varepsilon^{\frac{1}{2}}, \tag{53}$$

plus a boundary condition at $r = 1$. To derive this condition, we must, of course, go through a detailed boundary layer analysis for the side walls, in which we would reduce v in the core to zero at the walls by use of boundary layer correction functions, and we would insure that the volume flux through the side layers matched that in the core. A subsequent solution of the energy equations in the side layers would yield expressions for the radial heat flux due to boundary layer corrections to the temperature, which would then be used in writing condition (39). All this, of course, has been done in [3] and the analysis in this case would not differ, since we have assumed here that λ^*, $\gamma = 0(1)$. Hence, in this case the side layers would have closed circulations, double or triple structures, etc., which would yield a complicated expression for the boundary condition on θ. We assume, however, that this condition can be written in the linear form (contributions due to the closed circulation are neglected), and a generalization of this linear form then becomes

$$\frac{\partial \theta}{\partial r} = -\lambda^*\gamma^2 [\psi(r)\varepsilon^{-\frac{1}{2}}]\frac{\partial \theta}{\partial z}, \quad r = 1. \tag{54}$$

Here $\psi(r)$ is the value of the core stream function, which is determined by Elkman suction and given by (49). Thus (54) becomes

$$\frac{\partial \theta}{\partial r} = -\frac{\lambda^*\gamma^2}{2\sqrt{2}}[3\theta_t(r) + \theta_b(r)]\frac{\partial \theta}{\partial z}, \quad r = 1. \tag{55}$$

Equations (51), (52) and (55) now must be solved for θ. This is formally a non-linear problem, but we will be able to obtain a series solution.

The form of (52) suggests seeking a solution of the form

$$\theta(r,z) = z + \Phi(r). \tag{56}$$

With this assumed functionality, the problem for Φ then becomes a linear one. We note that

$$3\theta_t + \theta_b = 3(1 + \Phi) + (-1 + \Phi) = 2 + 4\Phi. \tag{57}$$

Hence, Φ satisfies

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr} = -2\frac{\lambda^*\gamma^2}{\sqrt{2}}\left[1 + \frac{1}{r}\frac{d}{dr}(r^2\Phi)\right]. \tag{58}$$

$$\frac{d\Phi}{dr} = -\frac{\lambda^*\gamma^2}{\sqrt{2}}(1 + 2\Phi), \quad r = 1. \tag{59}$$

As we will see, Φ can be determined to within an additive constant; this reflects the fact that original problem contained thermal conditions which involved only derivatives of the temperature. Thus, any constant plus a solution is also a solution. We take this constant to be zero, which means we may only determine differences in temperature using the solution generated for Φ.

We now define

$$s = \frac{\lambda^*\gamma^2}{\sqrt{2}} \tag{60}$$

and assume $\Phi(r)$ has a series in s, viz.

$$\Phi(r) = \sum_{n=1}^{\infty} s^n\Phi_n(r). \tag{61}$$

From (58) and (59) we obtain

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr} = -2, \tag{62}$$

$$\frac{d\Phi}{dr} = -1, \quad r = 1. \tag{63}$$

Also

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi_n}{dr} = -\frac{2}{r}\frac{d}{dr}(r^2\Phi_{n-1}), \tag{64}$$

$$\frac{d\Phi_n}{dr} = -2\Phi_{n-1}, \quad r = 1 \tag{65}$$

for $n = 2,3,\ldots$. Φ_1 can be determined as

$$\Phi_1 = -\frac{r^2}{2} + d, \tag{66}$$

where d is an arbitrary constant. For simplicity we take $d = 0$, with the understanding that the results are to be interpreted as difference between θ and some arbitrary temperature, or between θ at two different points.

The next few Φ_n are

$$\left.\begin{aligned}\Phi_2 &= r^4/4,\\ \Phi_3 &= -r^6/12,\\ \Phi_4 &= r^8/48,\\ \Phi^5 &= -r^{10}/240\end{aligned}\right\} \tag{67}$$

We now deduce the general relations,

$$\Phi_n = \frac{(-1)^n r^{2n}}{2n!}. \tag{68}$$

The proof is by induction. Assume (68) is true for n. Φ_{n+1} satisfies (64) and (65), viz.

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi_{n+1}}{dr} = -2\frac{1}{r}\frac{d}{dr}(r\Phi_n) = \frac{(-1)^{n+1}(2n+2)r^{2n}}{n!} \tag{69}$$

$$\frac{d\Phi_{n+1}}{dr} = \frac{(-1)^{n+1}}{n!}, \quad r = 1. \tag{70}$$

The solution to (69) and (70) yields

$$\Phi_{n+1} = \frac{(-1)^{n+1}}{(2n+2)}\frac{r^{2n+2}}{n!} = \frac{(-1)^{n+1}r^{2(n+1)}}{2(n+1)!} \tag{71}$$

which completes the proof. Thus, the function $\Phi(r)$ has the representation,

$$\Phi(r) = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda^*\gamma^2}{\sqrt{2}}\right)^n\frac{r^{2n}}{n!}. \tag{72}$$

The series is, of course, convergent for all $\lambda^*, \gamma^\circ, \lambda^*\gamma^2 < \infty$.

It is of interest to examine the solution evaluated at the top plate. Since θ is known only to within a constant, we arbitrarily set $\theta\,(0.1) = 1$. Then $\theta\,(r, 1)$, the top temperature at any radius, becomes

$$\theta(r,1) = 1 + \frac{1}{2}\sum \frac{(-1)^n s^n r^{2n}}{n!}. \qquad (73)$$

This function has a maximum at $r = 0$, and is concave downward, which means the top surface temperature decreases with increasing radius. This is caused by the fact that cool fluid from the bottom surface flows upward near the cylinder side wall and thus cools the top surface. From (33) and (53) it is seen that the parameters β^* and λ^* both increase with increasing heating rate Q. Thus we see from (60) and (73) that non-isothermaliteies of the top plate temperature increase with increasing heating rate; i.e. the difference in the top plate temperature at $r = 0$ and $r = 1$ increases with increasing Q. In experiments the actual top surface will be somewhere between a perfectly isothermal and a perfectly constant heat flux condition. Thus, (73) can be used to give a conservative *a priori* estimate of the temperature variation along a radius of the top plate.

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REFERENCES

1. S. ABELL, Heat transfer measurements in a right circular cylinder of fluid heated from above and rotated, M.S. thesis, University of Illinois, Urbana (1969).
2. T. H. DAVIES and W. D. MORRIS, Heat transfer characteristics of a closed loop rotating thermosyphon, *Third International Conference on Heat Transfer*, Vol. II, p. 172. A.I.Ch.E., New York (1966).
3. G. M. HOMSY and J. L. HUDSON. Centrifugally driven thermal convection in a rotating cylinder, *J. Fluid Mech.* **35**, 33 (1969).
4. G. M. HOMSY and J. L. HUDSON, Centrifugal convection and its effect on the asymptotic stability of a bounded rotating fluid heated from below, to be published in *J. Fluid Mech.*
5. J. L. HUDSON, Non-isothermal flow between rotating disks, *Chem. Engng Sci.* **23**, 1007 (1968).
6. R. LEMLICH, Natural convection to isothermal flat plate with a spatially nonuniform acceleration, *I/EC Fundamentals* **2**, 157 (1963).
7. R. LEMLICH and J. S. STEINKAMP, Laminar natural convection to an isothermal flat plate with a spatially varying acceleration, *A.I.Ch.E.Jl* **10**, 445 (1964).
8. J. E. LLANA, An experimental investigation of convection in a rotating cylinder heated from above, M.S. thesis, University of Illinois, Urbana (1968).
9. M. MANOFF and R. LEMLICH, Free convection to a rotating central plate in synchronously rotating surroundings with and without consideration of Coroilis forces, Eleventh National Heat Transfer Conference, A.I.Ch.E.—A.S.M.E., Minneapolis, 1969; **66**, 118 (1970). *Chem. Engng Prog. Symp. Ser.*
10. Y. MORI and W. NAKAYAMA, Convective heat transfer in rotating radial circular pipes, *Int. J. Heat Mass Transfer* **11**, 1027 (1968).
11. W. D. MORRIS, Laminar convection in a heated vertical tube rotating about a parallel axis, *J. Fluid Mech.* **21**, 453 (1965).
12. W. D. MORRIS, An experimental investigation of laminar heat transfer in a uniformly heated tube rotating about a parallel axis, Ministry of Technology, London, Aeronautical Research Council, C. P. No. 1055, (1969).
13. E. H. W. SCHMIDT, Heat transmission by national convection at high centrifugal acceleration in water cooled gas turbine blades, in I.M.E.–A.S.M.E. General Discussion on Heat Transfer, p. 361, Inst. Mech. Eng., London (1951).

TRANSFERT THERMIQUE D'UN FLUIDE CHAUFFÉ PAR LE HAUT DANS UN CYLINDRE TOURNANT

Résumé—On présente une analyse mathématique de la convection thermique conduite par centrifugation dans un cylindre qui tourne autour de son axe vertical. On fait deux extensions d'un travail antérieur. En premier on considère un flux thermique à travers la paroi latérale du cylindre et on détermine son effet sur le nombre de Nusselt au sommet et à la base du cylindre. En deuxième lieu on recherche une condition limite de flux thermique constant pour les surfaces horizontales.

WÄRMEÜBERGANG IN EINEM FLÜSSIGKEITSGEFÜLLTEN ROTIERENDEN ZYLINDER, DER VON OBEN BEHEIZT WIRD

Zusammenfassung—Die durch Zentrifugalkräfte bewirkte thermische Konvektion in einem Zylinder, der um seine vertikale Achse rotiert, wird mathematisch untersucht. Frühere Arbeiten werden in zweifacher Hinsicht erweitert. Erstens wird der Wärmestrom durch die Mantelfläche des Zylinders berücksichtigt und sein Einfluss auf die Nusselt-Zahl an der oberen und unteren Stirnfläche bestimmt. Zweitens wird die Randbedingung konstanten Wärmestroms an den Stirnflächen untersucht.

ТЕПЛООБМЕН ВО ВРАЩАЮЩЕМСЯ ЦИЛИНДРИЧЕСКОМ ОББЕМЕ ЖИДКОСТИ, НАГРЕВАЕМОМ СВЕРХУ

Аннотация—Представлен математический анализ тепловой конвекции в цилиндре, вращающемся вокруг вертикальной оси. Результаты предыдущей работы обобщены в двух направлениях. Во-первых, учтен тепловой поток через боковую стенку цилиндра и определено его влияние на число Нуссельта для верхнего и нижнего оснований цилиндра. Во-вторых, решена задача с постоянным тепловым потоком на горизонтальных поверхностях.